Primes of form $6^n + 1$

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Abstract: We pose various congruences on the integers of form $6^n + 1, n \in Z_+$, which may encourage younger number theorist to do research in number theory and settled new dimension in this field. We observed that there are only three prime numbers, namely $7, 37$, and 1297 of form $6^n + 1, n \in Z_+$, and no one Fermat numbers attains this form. Moreover, these integers end with 7, like Fermat numbers $F_n, n \geq 2$. Also, we discussed some congruences with number theoretic functions σ, φ and Möbious function μ.

Key Words: Congruences, Fermat Number, Number Theoretic Functions, Prime Number

Introduction: At the elementary level, the theory of numbers deals with properties of integers and especially with the positive integers $1, 2, 3, \dots \dots \dots$ (also known as the natural numbers), and primes are the nucleus of number theory. Primes and forms of integers have been studied for over two thousand years at the time of Euclid. The Euclid theorem provided various consequences, like that there are an infinite number of primes of form 4k+3 [3]. Multitude problems on primes and forms of integers are still open until now. The famous Goldbach conjecture ([4], [11]) for even integers, initially tells that every even integer $n > 2$ can be represented as a sum of two primes. The twin prime conjecture ([1], [6], [12], [17]), assertion that there are infinitely many primes that differ by 2. In the published paper [10] the author proved that all the integers of the form $p^6 + 6^p$, with prime $p \geq 2$ are composite. After the studies of above cited work and various literature referenced in ([2], [5], [7], [8], [9], [10], [11], [13], [14], [15], [16]) on various conjectures concerned with primes, the form of integers, numbers of special forms like Fermat numbers and number theoretic functions have great importance in the field of number theory. This study aims to provide some congruences on the integers of the form $6^n + 1, n \in Z_+$, with number theoretic functions σ, φ , Möbious function μ. Prominently we have that all numbers $6^n + 1, n \in Z_+$, end with 7 like the Fermat numbers $F_n = 2^{2^n} + 1, n \geq 2$.

Theorem (1): If the integer $n = 4k$, where k is an odd integer, then congruences $6^n + 1 \equiv 0(1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$ always hold.

Proof: Since $6^4 \equiv -1(mod\ 1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$. Consider for $n = 4k$, where k is odd, then clearly, the congruences $6^n + 1 \equiv 0(1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$ always hold.

Theorem (2): If the integer $n = 4.(2k)$, where k is an odd integer, then $6^n + 1 \equiv 0(17)$ and $6^4 \not\equiv -1(mod\ (17)^2)$ always hold.

Proof: Since $6^{4.(2k)} \equiv -1(mod\ 17)$. Consider for $n = 4.(2k)$, where k is odd, then clearly, the congruences $6^n + 1 \equiv 0(17)$ $6^n \not\equiv -1(mod\ (17)^2)$ always hold.

Remark (1): The chain of integers of the form $6^{4(2k+1)} + 1,\ k \in Z_0$ holds exactly one prime.

Remark (2): The chain of integers of the form $6^{4(2k+1)} + 1,\ k \in Z_0$ has no Fermat number, moreover these are composite numbers end with 7.

Remark (3): The chain of integers of the form $6^{8(2k+1)} + 1,\ k \in Z_0$ has no prime.

Conjecture (1): Each member of the chain of integers of the form $6^{4(2k+1)} + 1,\ k \in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma\left(6^{4(2k+1)} + 1\right) \equiv 0(mod\ 2^l), k \in Z_+, l \geq 2 \qquad (1.1.1)$$

$$\varphi\left(6^{4(2k+1)} + 1\right) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 4 \qquad (1.1.2)$$

$$\varphi\left[\varphi\left(6^{4(2k+1)} + 1\right)\right] \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2 \qquad (1.1.3)$$

$$\mu\left(\sigma\left(6^{4(2k+1)} + 1\right)\right) = 0, k \in Z_+ \qquad (1.1.4)$$

$$\mu\left(\varphi\left(6^{4(2k+1)} + 1\right)\right) = 0, k \in Z_0 \qquad (1.1.5)$$

$$\mu\left(\varphi\left[\varphi\left(6^{4(2k+1)} + 1\right)\right]\right) = 0, k \in Z_0 \qquad (1.1.6)$$

Proof: Consider theorem (1) and recall the definition of number theoretic functions $\sigma,\ \varphi$ and Möbious function μ. Then we find that the congruences (1.1.1) to (1.1.6) must hold.

Conjecture (2): Each member of the chain of integers of the form $6^{8(2k+1)} + 1,\ k \in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma\left(6^{8(2k+1)} + 1\right) \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2 \qquad (2.1.1)$$

$$\varphi\left(6^{8(2k+1)} + 1\right) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 4 \qquad (2.1.2)$$

$$\varphi\left[\varphi\left(6^{8(2k+1)}+1\right)\right] \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2 \qquad (2.1.3)$$

$$\mu\left(\sigma\left(6^{8(2k+1)}+1\right)\right) = 0, k \in Z_0 \qquad (2.1.4)$$

$$\mu\left(\varphi\left(6^{8(2k+1)}+1\right)\right) = 0, k \in Z_0 \qquad (2.1.5)$$

$$\mu\left(\varphi\left[\varphi\left(6^{8(2k+1)}+1\right)\right]\right) = 0, k \in Z_0 \qquad (2.1.6)$$

Proof: Consider theorem (2) and recall the definition of number theoretic functions σ, φ and Möbious function μ. Then we find that the congruences (2.1.1) to (2.1.6) must hold.

Theorem (3): If the integer $n = 4k$, where k is an odd integer, then congruences $6^n + 1 \equiv 0(1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$ always hold.

Proof: Since $6^4 \equiv -1(mod\ 1297)$, and $6^4 \not\equiv -1(mod\ (1297)^2)$. Consider for $n = 4k$, where k is odd**,** then clearly, the congruences $6^n + 1 \equiv 0(1297)$, and $6^n \not\equiv -1(mod\ (1297)^2)$ always hold.

Theorem (2): If the integer $n = 4.(2k)$, where k is an odd integer, then $6^n + 1 \equiv 0(17)$ and $6^4 \not\equiv -1(mod\ (17)^2)$ always hold.

Proof: Since $6^{4.(2k)} \equiv -1(mod\ 17)$. Consider for $n = 4.(2k)$, where k is odd**,** then clearly, the congruences $6^n + 1 \equiv 0(17)$ $6^n \not\equiv -1(mod\ (17)^2)$ always hold.

Remark (1): The chain of integers of the form $6^{4(2k+1)} + 1$, $k \in Z_0$ has exactly one prime.

Remark (2): The chain of integers of the form $6^{4(2k+1)} + 1$, $k \in Z_0$ has no Fermat number, moreover these are composite numbers end with 7.

Remark (3): The chain of integers of the form $6^{8(2k+1)} + 1$, $k \in Z_0$ has no prime.

Conjecture (1): Each member of the chain of integers of the form $6^{4(2k+1)} + 1$, $k \in Z_0$ must hold the following congruences with respect to number theoretic functions σ, φ and Möbious function μ.

$$\sigma\left(6^{4(2k+1)}+1\right) \equiv 0(mod\ 2^l), k \in Z_+, l \geq 2 \qquad (1.1.1)$$

$$\varphi\left(6^{4(2k+1)}+1\right) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 4 \qquad (1.1.2)$$

$$\varphi\left[\varphi\left(6^{4(2k+1)}+1\right)\right] \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2 \qquad (1.1.3)$$

$$\mu\left(\sigma\left(6^{4(2k+1)}+1\right)\right) = 0, k \in Z_+ \qquad (1.1.4)$$

$$\mu\left(\varphi\left(6^{4(2k+1)}+1\right)\right) = 0, k \in Z_0 \qquad (1.1.5)$$

$$\mu\left(\varphi\left[\varphi\left(6^{4(2k+1)}+1\right)\right]\right) = 0, k \in Z_0 \qquad (1.1.6)$$

Proof: Consider theorem (1) and recall the definition of number theoretic functions $\sigma,\ \varphi$ and Möbious function μ. Then we find that the congruences (1.1.1) to (1.1.6) must hold.

Conjecture (2): Each member of the chain of integers of the form $6^{8(2k+1)}+1,\ k\in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma\left(6^{8(2k+1)}+1\right)\equiv 0(mod\ 2^l3^m), k\in Z_0, l,m\geq 2 \qquad (2.1.1)$$

$$\varphi\left(6^{8(2k+1)}+1\right)\equiv 0(mod\ 2^l), k\in Z_0, l\geq 4 \qquad (2.1.2)$$

$$\varphi\left[\varphi\left(6^{8(2k+1)}+1\right)\right]\equiv 0(mod\ 2^l), k\in Z_0, l\geq 2 \qquad (2.1.3)$$

$$\mu\left(\sigma\left(6^{8(2k+1)}+1\right)\right)=0, k\in Z_0 \qquad (2.1.4)$$

$$\mu\left(\varphi\left(6^{8(2k+1)}+1\right)\right)=0, k\in Z_0 \qquad (2.1.5)$$

$$\mu\left(\varphi\left[\varphi\left(6^{8(2k+1)}+1\right)\right]\right)=0, k\in Z_0 \qquad (2.1.6)$$

Proof: Consider theorem (2) and recall the definition of number theoretic functions $\sigma,\ \varphi$ and Möbious function μ. Then we find that the congruences (2.1.1) to (2.1.6) must hold.

Theorem (3): If the integer $n=4k+1$, where $k\in Z_0$, then congruence $6^n+1\equiv 0(7)$, always hold, but the congruence $6^n+1\not\equiv 0(mod\ (7)^2)$ not necessarily hold.

Proof: Since $6^1\equiv -1(mod\ 7)$, and $6^1\not\equiv -1(mod\ (7)^2)$. Consider for $n=4k+1$, where $k\in Z_0$**,** then clearly, the congruences $6^n+1\equiv 0(mod\ 7)$), always hold. Since $n=4\times 5+1$ and $6^n+1\equiv 0(mod\ (7)^2)$ hold. This implies that $6^n+1\not\equiv 0(mod\ (7)^2)$ not necessarily hold.

Remark (4): The chain of integers of the form $6^{4k+1}+1,\ k\in Z_0$ has exactly one prime.

Remark (5): The chain of integers of the form $6^{4k+1}+1,\ k\in Z_0$ has no Fermat number, moreover these are composite numbers end with 7.

Remark (6): The chain of integers of the form $6^{7(2k+1)}+1,\ k\in Z_0$ satisfies the congruence $6^{7(2k+1)}+1\equiv 0(mod\ (7)^2)$.

Conjecture (3): Each member of the chain of integers of the form $6^{4k+1}+1,\ k\in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma(6^{4k+1}+1)\equiv 0(mod\ 2^l), k\in Z_0, l\geq 2 \qquad (3.1.1)$$

$$\varphi(6^{4k+1}+1)\equiv 0(mod\ 2^l3^m), k\in Z_0, l,m\geq 1 \qquad (3.1.2)$$

$$\varphi\left[\varphi\left(6^{4(2k+1)}+1\right)\right]\equiv 0(mod\ 2^l3^m), k\in Z_0, l\geq 1,\ m\geq 0 \qquad (3.1.3)$$

$$\mu\left(\sigma\left(6^{4(2k+1)}+1\right)\right)=0, k\in Z_0 \qquad (3.1.4)$$

$$\mu\left(\varphi\left(6^{4(2k+1)}+1\right)\right)=0, k\in Z_+ \qquad (3.1.5)$$

$$\mu\left(\varphi\left[\varphi\left(6^{4(2k+1)}+1\right)\right]\right)=0, k\in Z_+ \qquad (3.1.6)$$

Proof: Consider theorem (3) and recall the definition of number theoretic functions σ, φ and Möbious function μ. Then we find that the congruences (3.1.1) to (3.1.6) must hold.

Theorem (4): If the integer $n=4k+2$, where $k\in Z_0$, then congruences $6^n+1\equiv 0(37)$, and $6^n+1\not\equiv 0(mod\ (37)^2)$ always hold.

Proof: Since $6^2\equiv -1(mod\ 37)$, and $6^2\not\equiv -1(mod\ (37)^2)$. Consider for $n=4k+2,$ where $k\in Z_0$**,** then clearly, the congruences $6^n+1\equiv 0(mod\ 37)$), and $6^n+1\not\equiv -1(mod\ (37)^2)$ always hold.

Remark (6): The chain of integers of the form $6^{4k+2}+1,\ k\in Z_0$ holds exactly one prime.

Remark (7): The chain of integers of the form $6^{4k+2}+1,\ k\in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Conjecture (4): Each member of the chain of integers of the form $6^{4k+2}+1,\ k\in Z_0$ must hold the following congruences with respect to number theoretic functions σ, φ and Möbious function μ.

$$\sigma(6^{4k+2}+1)\equiv 0(mod\ 2^l), k\in Z_+, l\geq 2 \qquad (4.1.1)$$

$$\varphi(6^{4k+2}+1)\equiv 0(mod\ 2^l3^m), k\in Z_0, l,m\geq 2 \qquad (4.1.2)$$

$$\varphi[\varphi(6^{4k+2}+1)]\equiv 0(mod\ 2^l), k\in Z_0, l\geq 2 \qquad (4.1.3)$$

$$\mu\left(\sigma(6^{4k+2}+1)\right)=0, k\in Z_+ \qquad (4.1.4)$$

$$\mu\left(\varphi(6^{4k+2}+1)\right)=0, k\in Z_0 \qquad (4.1.5)$$

$$\mu(\varphi[\varphi(6^{4k+2}+1)])=0, k\in Z_0 \qquad (4.1.6)$$

Proof: Consider theorem (4) and recall the definition of number theoretic functions σ, φ and Möbious function μ. Then we find that the congruences (4.1.1) to (4.1.6) must hold.

Theorem (5): If the integer $n=4k+3$, where $k\in Z_0$, then congruence $6^n+1\equiv 0(mod\ 7)$, always hold but the congruence $6^n+1\not\equiv 0(mod\ (7)^2)$ not necessarily hold.

Proof: Since $6^3\equiv -1(mod\ 7)$, and $6^3\not\equiv -1(mod\ (7)^2)$. Consider for $n=4k+3,$ where $k\in Z_0$**,** then clearly, the congruence $6^n+1\equiv 0(mod\ 7)$), always hold. Since $n=7=4(1)+3$

then, $6^7 + 1 \equiv 0(mod\ (7)^2)$ hold (as in remark 6). Consequently, we have the congruence $6^n + 1 \not\equiv 0(mod\ (7)^2)$ not necessarily hold.

Remark (8): The chain of integers of the form $6^{4k+3} + 1,\ k \in Z_0$ holds no prime.

Remark (9): The chain of integers of the form $6^{4k+3} + 1,\ k \in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Conjecture (5): Each member of the chain of integers of the form $6^{4k+3} + 1,\ k \in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma(6^{4k+3} + 1) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2 \qquad (4.1.1)$$

$$\varphi(6^{4k+3} + 1) \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2 \qquad (4.1.2)$$

$$\varphi[\varphi(6^{4k+3} + 1)] \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2 \qquad (4.1.3)$$

$$\mu\left(\sigma(6^{4k+3} + 1)\right) = 0, k \in Z_+ \qquad (4.1.4)$$

$$\mu\left(\varphi(6^{4k+3} + 1)\right) = 0, k \in Z_0 \qquad (4.1.5)$$

$$\mu(\varphi[\varphi(6^{4k+3} + 1)]) = 0, k \in Z_0 \qquad (4.1.6)$$

Proof: Consider theorem (5) and recall the definition of number theoretic functions $\sigma,\ \varphi$ and Möbious function μ. Then we find that the congruences (4.1.1) to (4.1.6) must hold.

Remark (8): The chain of integers of the form $6^{6k+3} + 1,\ k \in Z_0$ satisfies the congruence $6^{8k+7} + 1 \equiv 0(mod\ 31)$.

Remark (9): The chain of integers of the form $6^{6k+3} + 1,\ k \in Z_0$ holds no Fermat number, moreover these are composite numbers end with 7.

Conjecture (6): Each member of the chain of integers of the form $6^{6k+3} + 1,\ k \in Z_0$ must hold the following congruences with respect to number theoretic functions $\sigma,\ \varphi$ and Möbious function μ.

$$\sigma(6^{6k+3} + 1) \equiv 0(mod\ 2^l), k \in Z_0, l \geq 8 \qquad (5.1.1)$$

$$\varphi(6^{6k+3} + 1) \equiv 0(mod\ 2^l 3^m), k \in Z_0, l, m \geq 2 \qquad (5.1.2)$$

$$\varphi[\varphi(6^{6k+3} + 1)] \equiv 0(mod\ 2^l), k \in Z_0, l \geq 2 \qquad (5.1.3)$$

$$\mu\left(\sigma(6^{6k+3} + 1)\right) = 0, k \in Z_+ \qquad (5.1.4)$$

$$\mu\left(\varphi(6^{6k+3} + 1)\right) = 0, k \in Z_0 \qquad (5.1.5)$$

$$\mu(\varphi[\varphi(6^{6k+3} + 1)]) = 0, k \in Z_0 \qquad (5.1.6)$$

Proof: Consider remark (8) and recall the definition of number theoretic functions $\sigma,\ \varphi$ and Möbious function μ. Then we find that the congruences (5.1.1) to (5.1.6) must hold.

Conclusion: We see that all numbers of form $6^n + 1, n \in Z_+$, end with 7 like the Fermat numbers $F_n = 2^{2^n} + 1, n \geq 2$, but no Fermat numbers reach such form. Only three primes 7, 37, and 1297 exist in this form. Mostly members of the string of the form $6^n + 1, n \in Z_+$, of integers hold beautiful congruences with number theoretic functions σ, φ and Möbious function μ.

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